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Harmonic resolution as a holographic quantum number

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Abstract: The Bekenstein bound takes the holographic principle into the realm of flat space, promising new insights on the relation of non-gravitational physics to quantum gravity. This makes it important to obtain a precise formulation of the bound. Conventionally, one specifies two macroscopic quantities, mass and spatial width, which cannot be simultaneously diagonalized. Thus, the counting of compatible states is not sharply defined. The resolution of this and other formal difficulties leads naturally to a definition in terms of discretized light-cone quantization. In this form, the area difference specified in the covariant bound converts to a single quantum number, the harmonic resolution K. The Bekenstein bound then states that the Fock space sector with K units of longitudinal momentum contains no more than $\exp(2\pi^2 K)$ independent discrete states. This conjecture can be tested unambiguously for a given lagrangian, and it appears to hold true for realistic field theories, including models arising from string compactifications. For large K, it makes contact with more conventional but less well-defined formulations.

Keywords: Black Holes in String Theory, M(atrix) Theories, Models of Quantum Gravity.

Contents

1. Introduction

It was recently shown [1] that the Bekenstein bound [2, 3] can be derived from a generalized form [4] of the covariant bound on the entropy of lightsheets [5]. This derivation becomes exact for weakly gravitating systems in flat space. It yields

$$S \leq \frac{\pi M a}{\hbar}, \tag{1.1}$$

where S is the entropy of a matter system with energy up to M and spatial width up to a.

The width a is the distance between any pair of parallel planes clamping the system. For example, if the system fits into a rectangular box, a can be taken to be its shortest side. Hence, (1.1) is actually stronger than Bekenstein's original version, $S \leq \pi M d/\hbar$, which referred to the diameter d of the smallest sphere capable of enclosing the system.

Earlier derivations of the Bekenstein bound applied the generalized second law of thermodynamics to systems that are slowly lowered into a black hole, prompting a controversy about the role of quantum effects and other subtleties arising in this rather nontrivial process. The new derivation of Bekenstein's bound is immune to such difficulties as it takes place in the benign environment of flat space and involves no accelerations.

Most importantly, the new derivation identifies the Bekenstein bound as a special limit of the covariant bound [5], a conjectured empirical pattern underlying the holographic principle [6, 7, 8]. This limit is both intriguing and especially simple because it applies to weakly gravitating systems. It can be tested entirely within quantum field theory, without inclusion of gravity. Moreover, as we will argue in a separate publication [9], the absence of Newton's constant in the Bekenstein bound signifies that key aspects of quantum mechanics can be derived from classical gravity together with the holographic relation between information and geometry. Hence, it will be of great importance to obtain a completely well-defined and unambiguous formulation of the Bekenstein bound.

Of course, our understanding of the Bekenstein bound is no worse than that of the covariant bound. However, for the purposes of the covariant bound [5], the entropy S can be satisfactorily defined as the logarithm of the number of independent quantum states compatible with assumed macroscopic conditions. Such conditions, at the very least, are always implicit in the specification of the area appearing on the right hand side of the bound. Because this area must be large in Planck units, the bound can only be challenged by systems with large entropy. This is why in most situations that are of interest for testing the bound, such as in cosmology and for macroscopic isolated systems, thermodynamic approximations are valid and the value of S is not sensitive to subtleties (such as the definition of "compatible").

By contrast, the Bekenstein bound and (by extension) the generalized covariant bound [4] are most readily challenged by systems with few quanta. This makes them sensitive to the details of the entropy definition. Indeed, various authors, using inequivalent definitions, have reached different conclusions about the validity of the Bekenstein bound [3, 10]–[25]. Our point of view is that any concise formulation that renders the Bekenstein bound well-defined, nontrivial, and empirically true will capture a potentially interesting fact about Nature. Moreover, it may have implications in the general context of the covariant bounds, and it may help us sharpen their definitions as well. Hence, we use a variety of considerations to seek such a definition.

We have recently argued [26] that S should be defined microcanonically, as the logarithm of the number of exact eigenstates of the hamiltonian with energy $E \leq M$ and spatial width no greater than a. In particular, only bound states (states with discrete quantum numbers) contribute to the entropy, since scattering states have infinite size, and the only alternative — ad hoc imposition of boundary conditions — can be shown to trigger violations of the bound.

This definition, summarized in section 2, is quite successful heuristically. However, it does retain one annoying ambiguity (section 3): The spatial width of a quantum bound state is not sharply defined. Though wavefunctions tend to be concentrated in finite regions, they do not normally have strictly compact support. For example, there is a tiny but nonzero probability to detect the electron a meter away from the proton in the ground state of hydrogen. Of course, the width can be assigned some rough value corresponding to the region of overwhelming support. But this forces us to answer the sharp question of whether or not a given state contributes to S by an inherently ambiguous decision whether the state can be considered to have width smaller than a.

This problem is compounded by a practical difficulty: the hamiltonian methods required for the computation of bound states are often intractable in quantum field theory. Moreover, we show that aspects of the formulation of Bekenstein's bound have no justification from the point of view of its more recent derivation (which we regard as its real origin). Specifically, we criticize that not one but two macroscopic parameters are specified, and that these parameters act only to limit, but not to fix, the mass and size of allowed states.

In section 4 we systematically develop modifications designed to resolve these problems. Guided by the derivation of Bekenstein's bound from the GCEB, we construct a Fock space of states directly on the light-sheet via light-cone quantization. This allows us to identify the surplus parameter in the bound as a pure gauge choice. Moreover, light-cone quantization famously facilitates the use of hamiltonian methods in quantum field theory. Two other problems, most notably the width ambiguity, remain.

However, in the light-cone frame, one can adopt a different gauge which fixes the maximum width of states instead of the total momentum. In this gauge it becomes possible to identify the light-sheet periodically on a null circle of fixed length. Quantization on this compactified background is known as discretized light-cone quantization (DLCQ). One of its simplifying features, much exploited in QCD calculations, is that the Fock space breaks up into distinct sectors preserved by interactions, so that the hamiltonian can be diagonalized in each of them separately. Each sector is characterized by the number of units of momentum along the null circle, K.

The integer K (the "harmonic resolution") subsumes the two macroscopic parameters M and a. The Fock space contains a finite number of bound energy eigenstates for each integer K. The entropy S is defined to be the logarithm of that number, and the Bekenstein bound takes the form

$$S \leq 2\pi^2 K \tag{1.2}$$

in DLCQ.

In this form the bound is unambiguously defined and free of all of the earlier problems we had identified. The width of quantum states is imposed by the compactification. The bound manifestly contains only one parameter, K, to which all contributing microstates correspond exactly. Because of the further simplification of the Fock space structure, DLCQ is even better suited for finding bound states than ordinary light-cone quantization. Thus, all of the shortcomings we identified are resolved.

An interesting question is whether the refined definition of entropy developed here for flat space can be lifted back to the more general environment in which the covariant bounds operate. Here we hit upon a puzzle. Since our prescription involved compactifying a null direction (or equivalently, demanding periodicity), it does not naturally extend to curved space. When the contraction of a light-sheet cannot be neglected, its generators cannot be periodically identified.

It is intriguing that by demanding a completely unambiguous formulation of the Bekenstein bound, and taking seriously that entropy bounds are tied to null surfaces, one is naturally led to the framework of discretized light-cone quantization. Traditionally, DLCQ has been considered no more than a convenient trick for simplifying numerical calculations in QCD. More recently, it appeared in a more substantial role in the context of the Matrix model of M-theory [27, 28]. Its independent emergence in the context of entropy bounds suggests that DLCQ may have wider significance. If this were the case, then the spectra at finite harmonic resolution may have a direct physical interpretation.

2. Defining entropy

We will now discuss our starting point for the definition of entropy in the Bekenstein bound. In ref. [26], a combination of formal and empirical arguments led us to adopt a definition in which only bound states contribute to the entropy. That is,

$$S(M,a) \equiv \log \mathcal{N}(M,a)\,, \tag{2.1}$$

where $\mathcal{N}$ is the number of independent eigenstates of the hamiltonian, with energy eigenvalue

$$E \leq M\,, \tag{2.2}$$

total three-momentum eigenvalue

$$\mathbf{P} = 0\,, \tag{2.3}$$

and with spatial support over a region of width no larger than a. The bound takes the form

$$S(M,a) \leq \frac{\pi M a}{\hbar}\,. \tag{2.4}$$

We now summarize the arguments for this formulation.

The restriction to exact energy eigenstates is motivated not only by the conceptual clarity of the microcanonical ensemble [29]. The bound explicitly contains the mass (and not, for example, a temperature) on the right hand side. Thus, energy is a natural macroscopic parameter to which microstates must conform, via eq. (2.2). Moreover, in the derivation of the Bekenstein bound from the GCEB, the mass enters explicitly as the source of focussing of light rays; no other thermodynamic quantities appear. There are also empirical reasons: alternative definitions (involving, for example, ensembles at fixed temperature [30] or mixed states constructed from states other than energy-eigenstates [24]) were found to lead to violations of the bound.

Obviously, the bound is nontrivial only for states with finite width a. Yet, we expect energy eigenstates to be spread over all of space. Indeed, for states which are also eigenstates of the total spatial momentum, the overall phase factor corresponding to the total momentum signifies a complete delocalization of the center of mass. This conundrum can be resolved by integrating over all spatial momenta. In practice, it is simpler to continue to work with eigenstates of the full four-momentum, but to factor out and ignore the center of mass coordinates. We demand only that the wavefunction have finite spreading in the position space relative to the center of mass.

In free field theory, however, the constituents of multi-particle states are not bound, but are delocalized relative to each other. Therefore, the bound is essentially trivial in free field theory: multi-particle states have infinite spatial width and do not contribute to the entropy. One way of enforcing finite width would be to impose rigid boundary conditions by fiat. This type of prescription leads to apparent violations of the bound [26]. In fact it is physically incomplete, because the material enforcing the assumed boundary conditions (for example, a capacitor with enough charge carriers [25]) is not included in the mass and width.

Therefore the finite width requirement can be satisfied only if interactions are properly included from the start. Real matter systems localize themselves by the mutual interactions of constituent particles. In situations where the bound has nontrivial content, this implies that $\mathcal{N}$ counts energy eigenstates with finite spatial width. In other words, the only contribution to $\mathcal{N}(M, a)$ comes from bound states, which have no continuous quantum numbers. This statement can be thought of as a precise version of Bekenstein's requirement [13, 25] that only "complete systems" be considered.

In ref. [26] this conclusion was supported by an empirical analysis. We began with a free scalar and imposed boundary conditions by fiat. Then we estimated the mass of the materials necessary for enforcing them. Lower bounds on the mass of these additional components were obtained in two different ways, using different necessary conditions for localization. We found that only one such condition — the need for interactions so that particles can bind — gives rise to extra energy sufficient to uphold the bound in each of a diverse set of problematic examples [26]. The study of incomplete systems thus informs us that interactions should be key to the definition of a complete system.

Each bound state gives rise to a continuous three-parameter set of energy eigenstates related by boosts. Since these states all represent the same physical state in different coordinate systems, we should not count them separately, but mod out by overall boosts. Usually this is done implicitly by picking a Lorentz frame and declaring it to be a rest frame of the "system". The condition (2.3) formalizes this requirement by requiring that the spatial components of the total four-momentum of each allowed state must vanish.

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3. Problems of the present formulation

The form (2.1), (2.4) is an improvement over less precise (or obviously incorrect) statements of the Bekenstein bound, but it is still not satisfactory. We will now identify some of its shortcomings. We list four problems: one ambiguity, one practical difficulty, and two formal shortcomings.

3.1 Width ambiguity

This is the most pernicious problem because it renders the entropy S manifestly ambiguous and appears to invite violations of the bound.

Energy eigenvalues are precisely defined, but the spatial width of a bound energy eigenstate is an ambiguous concept. In order to define a width at all, one has to ignore the overall phase factor corresponding to the complete delocalization of the center of mass. One can ask, however, about the spreading of the wave function in the remaining position space relative to the center of mass. As we discussed in section 2, this spreading is infinite for scattering states, but finite for bound states. However, wave functions of bound states do not normally have strictly compact support in this position space; generically, one expects at least exponential tails outside any finite region. How are we to define the width of such a state precisely?

One possibility is to call a state localized to a spatial region $\mathcal{V}$ of width a if it is unlikely to find any of its constituents outside of $\mathcal{V}$, i.e., if the normalized wavefunction obeys a condition of the form $1 - \int_{\mathcal{V}} |\Psi|^2 < \eta$. But this introduces an arbitrary parameter $\eta \ll 1$, to which the integer $\mathcal{N}(M, a)$ is necessarily somewhat sensitive. A second possibility, which we will also dismiss shortly, is to modify the Fock space construction by considering theories on flat backgrounds in which one spatial direction is compactified on a circle of length a.

The problem is particularly serious for single particle states. Multiparticle bound states can be expanded into superpositions of product states. The corresponding position space functions yield a spatial width relative to the center of mass. The center of mass itself is always completely delocalized for momentum eigenstates, and the corresponding overall phase factor must be ignored to get a finite answer. But by this definition, single particle states of free fields would be assigned zero spatial size, leading to obvious violations of the bound.

3.2 Inadequacy of hamiltonian methods

Bound states are exceedingly difficult to find exactly in quantum field theory. In strongly coupled theories even the vacuum is highly nontrivial and differs significantly from the Fock space vacuum of the free theory. For this reason, hamiltonian dynamics is usually abandoned in favor of a lagrangian formulation that lends itself to the computation of scattering amplitudes, but not of bound states. This does not necessarily signal a fundamental problem, but it does appear to render the verification of the bound intractible precisely for the theories in which it is most interesting.

3.3 Extra macroscopic parameters

This and the following objection are related to the derivation of Bekenstein's bound from the GCEB [26]: We will show that the statement of the Bekenstein bound in section 2 is inconsistent with its covariant origin.

The entropy S in the GCEB is associated with matter systems whose energy focuses the cross-sectional area of certain light-rays by $\Delta A = A - A'$ [26]. Hence, ΔA is the natural "macroscopic parameter" held fixed while counting compatible states. The derivation of the Bekenstein bound from the GCEB converts this area difference into the product Ma. This suggests that the entropy in the Bekenstein bound should not be obtained by specifying mass and width separately. Only their product, Ma, should be held fixed as a single macroscopic parameter, because only this product matters as far as the amount of focussing is concerned.

To emphasize this, let us define a new dimensionless variable:

$$K = \frac{Ma}{2\pi\hbar}, \tag{3.1}$$

where a factor of 2π has been inserted for later convenience. We define $\mathcal{N}(K)$ as the number of bound states with vanishing total momentum, whose rest mass times spatial width does not exceed $2\pi K\hbar$. With $S(K) \equiv \log \mathcal{N}(K)$, the bound takes the form

$$S(K) \leq 2\pi^2 K. \tag{3.2}$$

Technically, this reformulation remedies our objection. However, eq. (3.2) appears to lead to a messy picture, in which states of hugely different energy ranges and spatial sizes all contribute to the entropy for given K. In particular, eq. (3.2) rules out the possibility of resolving the width ambiguity (section 3.1) by formally compactifying on a spatial circle of fixed length.

3.4 Excess parameter range

We have defined $\mathcal{N} = e^S$ as the number of states with energy and width *up to* M and a [or with a product of energy and width up to K, in the modification (3.2)]. However, the derivation of the Bekenstein bound from the GCEB [26] does not actually support the inclusion of states with less energy or smaller size. Whether two surfaces, or their areas A and A', or only the area difference $A - A' \sim Ma \sim K$ is held fixed: in either case, only those states should be admitted whose energy and width correspond precisely to K. But this would render the bound trivial: except for accidental exact degeneracies, the number of states corresponding precisely to the specified parameters would be either zero or one. Moreover, such a formulation would exacerbate the earlier problem of width ambiguity.

4. DLCQ as a precise definition of entropy

4.1 Assessment

Two of the problems we have listed concern the fact that parts of our definition of entropy are hard to justify from the point of view of the GCEB. As we turn to remedy the situation and reformulate the Bekenstein bound, it is therefore appropriate to look to its covariant heritage for clues. Indeed, there is a crucial aspect of the covariant bounds that the form (2.1), (2.4) of Bekenstein's bound fails to capture: The GCEB refers to quantum states on a portion of a light-sheet [5, 4]. That is, it applies to a hypersurface with two spatial and one null dimension, as opposed to a spatial volume.

Because of the restriction to energy eigenstates, the time at which the bound is evaluated is irrelevant, but this does not mean that the proper definition of the entropy is equally transparent in all frames. In section 2, it was implicit that the Fock space is constructed by equal-time quantization of the field theory in the usual manner; then the energy eigenstates conforming to the specified macroscopic parameters are counted. But why artificially introduce an arbitrary time coordinate, when the light-sheet L already picks out a (null) slicing of spacetime?

It is far more natural to regard L itself as a time slice, to construct a Fock space of states on it, and to count the number of bound states directly on the light-sheet. The derivation of Bekenstein's bound becomes exact in the limit $G \to 0$, i.e., when all curvature radii induced by matter are much larger than the matter system itself [26]. In this limit, L does not contract and constitutes a *front* [31]: a null hyperplane in Minkowski space, given for example by $t + x =$ const. The construction of a Fock space on this hypersurface is known as front-form quantization (and, less appropriately but more frequently, as "light-cone quantization" or quantization in the infinite momentum frame) [32]–[35]. We will briefly review the formalism; then we will show how it addresses the problems we have identified.

4.2 Light-cone quantization

With the coordinate change

$$x^+ = \frac{t+x}{\sqrt{2}}\,, \qquad x^- = \frac{t-x}{\sqrt{2}}\,, \tag{4.1}$$

the metric of Minkowski space is

$$ds^2 = 2dx^+dx^- - (x^\perp)^2\,, \tag{4.2}$$

where $x^\perp$ stands for the transverse coordinates, y and z. The total four-momentum has components

$$P^+ = P_- = \frac{E+P^x}{\sqrt{2}}\,, \qquad P^- = P_+ = \frac{E-P^x}{\sqrt{2}}\,, \tag{4.3}$$

and transverse components $P^\perp = (P^y, P^z)$.

In light-cone quantization, x^+ plays the role of time, whereas the longitudinal coordinate x^- replaces the third spatial variable. The momentum compontent P_+ plays the role of a hamiltonian; P_- is called the longitudinal momentum. Both quantities are positive definite.

One-particle states are created by acting on the vacuum with operators $a^\dagger_{k_-k_\perp}$ corresponding to modes

$$u_{k_-k_\perp} \sim \exp(ik_+x^+ + ik_-x^- + ik_\perp x_\perp)\,. \tag{4.4}$$

(We suppress extra indices distinguishing different fields and additional scalar, vector, or matrix factors for normalization and components.) After integrating out all zero-modes[1] the one-particle longitudinal momentum, k_-, is strictly positive. The one-particle light-cone energy is given by

$$k_+ = \frac{m^2 + k_\perp^2}{2k_-}\,. \tag{4.5}$$

[1] This may generate additional potential terms which capture nontrivial aspects of the structure of the vacuum such as symmetry breaking [36].

Because of the positivity of k_-, and because P_- is conserved, all interaction terms contain at least one annihilation operator. There are no terms like $a^\dagger_{k_{-,1}} a^\dagger_{k_{-,2}} a^\dagger_{k_{-,3}}$. Hence, there are no radiative corrections to the vacuum, and the Fock space can be constructed just as in the free theory. (This constitutes one of the chief advantages of light-cone quantization.)

As usual, the Fock space consists of products of one-particle states obtained by acting several times with creation operators. The free part of the hamiltonian takes the form

$$H_{\text{free}} = \int_0^\infty dk_- \int d^2k_\perp \; k_+ \; a^\dagger_{k_- k_\perp} a_{k_- k_\perp} \,. \tag{4.6}$$

Bound states are eigenstates of the full hamiltonian with no continuous quantum numbers. Bound states can be represented by wavefunctions that describe their decomposition into the Fock space states. Thus, light-cone quantization permits a Lorentz-invariant constituent interpretation of bound states even in strongly coupled theories such as QCD [37].

4.3 The Bekenstein bound in front form

Let us now formulate Bekenstein's bound in the light-cone frame. We go back to its derivation from the covariant bound, from which the Bekenstein most directly emerges in the covariant form [1]

$$S \le \frac{\pi (P_a k^a) \Delta\alpha}{\hbar} \,. \tag{4.7}$$

Here α is an affine parameter along the generators of the light-sheet L, and $\Delta\alpha$ is the length of the partial light-sheet occupied by the matter system in question, i.e., the "affine width" of the system as seen by a set of parallel light-rays. $k^a = dx^a/d\alpha$ is the future-directed null vector tangent to the light-sheet, and P_a is the total four-momentum [1] of the matter system.[2]

In ref. [1] this expression was further simplified by specializing to an arbitrary rest frame. Then the spatial momentum components vanish, the affine width becomes ordinary spatial width, and one obtains eq. (1.1). We will now express eq. (4.7) in light-cone coordinates instead.

We take the light-sheet L to be the null hypersurface $x^+ = \text{const}$.[3] In the form (4.7), the Bekenstein bound is invariant under rescaling of the affine parameter. We choose $\alpha = x^-$, so that the affine width is

$$\Delta\alpha = \Delta x^- \,. \tag{4.8}$$

[2]Here P^a is defined so that its components correspond to the energy and the physical momentum components, e.g., $P^x > 0$ for a particle moving in the positive x-direction. We choose the metric signature $(+ - - -)$ used in most of the field theory literature on light-cone quantization. By contrast, in ref. [1] the usual $(- + + +)$ convention was used, and $-P^a$ stood for the physical energy-momentum four-vector, so eq. (4.7) took the same form.

[3]This differs from ref. [1], where the light-sheet was the hypersurface $t - x = 0$. The change is made to conform to the usual choice of surfaces of constant time in the light-cone quantization literature.

Then the tangent vector $k^a = dx^a/d\alpha$ has components $(0,1,0,0)$ in the metric (4.2). The expression $(P_a k^a)$ is thus simply the longitudinal momentum P_-, and eq. (4.7) takes the form

$$S \le \frac{\pi P_- \Delta x^-}{\hbar} \,. \tag{4.9}$$

Note that the light-cone energy, P_+, does not appear in the bound. It is also independent of the value of the transverse momenta, $P_\perp$. P_- and Δx^- aquire opposite factors under boosts, so that the product $P_- \Delta x^-$ remains invariant. Indeed, boosts can be interpreted simply as a rescaling of the affine parameter. In this sense, manifest Poincaré invariance, though spoiled when specializing to a spatial frame, is nearly retained by the front form expression (4.9). That is, Poincaré transformations have no effect on eq. (4.9) except for rescalings of the affine parameter.

So far, we have only expressed the bound in a new coordinate system. Next, we turn to the question of defining S in the light-cone frame. Here we reap some benefits that allow us to address two of the four shortcomings listed in section 3.

The direct analogue of the prescription (2.1) would be to specify two macroscopic parameters, P_- and Δx^-, and to define the entropy by

$$S = \log \mathcal{N}_{LCQ}(P_-, \Delta x^-) \,, \tag{4.10}$$

where $\mathcal{N}_{LCQ}(P_-, \Delta x^-)$ is the number of eigenstates of the total four-momentum, whose longitudinal momentum and affine width do not exceed the specified parameters.

We also require an analogue of the gauge condition, eq. (2.3), to ensure that states related by overall boosts are counted only once. This condition can be adapted to the light-cone frame by fixing those components of the four-momentum which are canonically treated as spatial, namely P_- and $P_\perp$. The transverse momenta can be set to zero as before:

$$P_\perp = 0 \,, \tag{4.11}$$

which projects out states related by transverse boosts. However, one of the peculiarities of the light-cone frame is that the longitudinal momentum is strictly positive for massive states. It cannot be gauge-fixed to zero by boosting. In order to mod out by longitudinal boosts, P_- must instead be set to an arbitrary positive constant:

$$P_- = \text{const} \,. \tag{4.12}$$

This exposes the "macroscopic parameter" P_- specified in the definition (4.10) of the entropy as a gauge choice. Only the width Δx^- is a physical parameter. Thus, when the bound is formulated in the light-cone frame, the existence of only one macroscopic parameter is manifest, and the objection in section 3.3 is resolved.

In section 3.2 we objected that hamiltonian methods, which are crucial to the identification of bound states and thus to our definition of entropy, are impractical and hardly used in quantum field theory. But in fact, light-cone quantization facilitates the use of hamiltonians considerably. For example, the light-cone hamiltonian P_+, unlike the energy E, can be evaluated from the other four-momentum components without use of a square root; see eqs. (4.5) and (4.6). Moreover, the ground state of the free theory is also a ground state of the interacting hamiltonian. For these and other reasons, light-cone quantization has emerged as a leading tool for finding the spectrum and wavefunctions of bound states in QCD and other interacting theories [37]. Although we were guided to the front form by a different consideration (the covariant pedigree of Bekenstein's bound), we thus find that light-cone quantization is custom-designed for the task of defining the relevant entropy.

4.4 Resolving the width ambiguity by compactification

Having succeeded in resolving two of the four problems identified in section 3, we now turn to the two remaining difficulties — in particular, the dreaded width ambiguity.

Let us rewrite eq. (4.9) in the manifestly Lorentz-invariant form:

$$S(K) \leq 2\pi^2 K\,, \tag{4.13}$$

where K is an arbitrary non-negative number specified as a macroscopic parameter. In the light-cone frame, K is given by

$$K = \frac{P_- \Delta x^-}{2\pi\hbar}\,. \tag{4.14}$$

Once P_- is gauge-fixed to a constant, specification of the parameter K is equivalent to specification of Δx^-.

Its manifest boost invariance allows us to think of the front form of Bekenstein's bound in two ways. In both versions, K is the single macroscopic parameter. Until now we have chosen to gauge-fix P_- and obtain a bound for every positive K, concerning the entropy of states whose maximal width depends on K as

$$\Delta x^- = \frac{2\pi K\hbar}{P_-}\,. \tag{4.15}$$

An alternative, equivalent option is to gauge-fix Δx^- but leave P_- to be determined by

$$P_- = \frac{2\pi K\hbar}{\Delta x^-}\,. \tag{4.16}$$

This also yields a bound for every positive K, concerning the entropy of states of fixed width but K-dependent maximal longitudinal momentum.

Both pictures yield the same number of states, because every physical state allowed for a given value of K is mapped to a boosted version of itself when the picture is changed. But the second picture, in which the width is gauge-fixed, serves as a point of departure for a new formulation of the Bekenstein bound which circumvents the ambiguity of the width of a quantum state.

We may now directly enforce a kind of width limit on quantum states simply by compactifying the x^- direction on a light-like circle of affine length Δx^-. This contrasts with the rest frame, in which no such unique compactification is possible, because the spatial width a is still variable even after specifying Ma and gauge-fixing the three-momentum to zero. Because Δx^- can be gauge-fixed, and can be fixed to the same value independently of K, we can consistently compactify on a fixed null circle.

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Note that a prescription that involves compactification is a genuine modification of the bound.[4] It changes the spectrum, especially at small values of K. The finite size of the longitudinal direction means that the distinction between bound states and scattering states can only be based on the bahavior in the transverse directions. If this prescription is the correct formulation of the Bekenstein bound, then the application of the bound to real systems will require choosing K so large that the effects of compactification are negligible. In any case, the ambiguity of defining the spatial width of a quantum states forces a compactified formulation upon us.

This presents us with the task of constructing a Fock space of states on a light front with periodic boundary conditions. Fortunately, this formalism is well understood; indeed, discretized light-cone quantization [38, 39] is one of the chief tools for calculating bound states in QCD [37]. Let us briefly review the key elements.

4.5 Discretized light-cone quantization

Compactification of the x^- direction discretizes all longitudinal momenta, which must be integer multiples of $2\pi\hbar/\Delta x^-$. In particular, the parameter

$$K = \frac{P_- \Delta x^-}{2\pi\hbar} \tag{4.17}$$

is now a non-negative integer called the harmonic resolution. The correspondingly modified Fock space construction is called discrete light-cone quantization (DLCQ). One-particle states still correspond to modes

$$u_{k_- k_\perp} \sim \exp(ik_+ x^+ + ik_- x^- + ik_\perp x^\perp)\,, \tag{4.18}$$

but now their longitudinal momentum is discrete:

$$k_- = \frac{2\pi n\hbar}{\Delta x^-}\,, \tag{4.19}$$

where n is a positive integer.

Because P_- is conserved by interactions, the Fock space decomposes into an infinite number of *inequivalent* sectors, one for each nonnegative integer K. Note that the one-particle states have positive, quantized longitudinal momenta, which must add up to $2\pi K\hbar/\Delta x^-$ in the K-th sector. This makes the Fock space sectors of DLCQ comparatively simple. For example the $K = 1$ sector can only contain one-particle states, all of which have $k_- = 2\pi\hbar/\Delta x^-$.

4.6 The Bekenstein bound in DLCQ form

Given a field theory in discretized light-cone quantization, let $\mathcal{N}_{\mathrm{DLCQ}}(K)$ be the number of bound states in the sector of the Fock space with harmonic resolution K. By bound

[4] It is thus a more radical step than merely going to the light-cone frame, which is merely better adapted but physically equivalent to ordinary Lorentz frames.

states we mean those states in the spectrum of the hamiltonian P_+ which are discrete up to overall boosts. We define the entropy

$$S_{\text{DLCQ}}(K) = \log \mathcal{N}_{\text{DLCQ}}(K)\,. \tag{4.20}$$

The Bekenstein bound in DLCQ form is the conjecture that

$$S_{\text{DLCQ}}(K) \leq 2\pi^2 K\,. \tag{4.21}$$

For completeness we summarize the gauge conditions again. Previously they corresponded to fixing the total momentum, as in eq. (2.3), or eqs. (4.11) and (4.12). In the DLCQ formulation, we still must set the transverse momenum components to a fixed value; for example,

$$P_\perp = 0\,. \tag{4.22}$$

We no longer gauge-fix P_-; that is replaced by picking an arbitrary but fixed compactification length Δx^-. Note that the spectrum depends trivially on Δx^-, and the entropy (4.20) in the sector K does not depend on Δx^- at all.

Let us summarize how the problems listed in section 3 have been resolved by formulating the Bekenstein bound in DLCQ. The problem of defining the width of quantum states (section 3.1) is circumvented, because width enters only implicitly through the fixed compactification scale, to which all states conform by construction. The entropy S is defined unambiguously by the specification of only a single parameter, K, which corresponds to the area difference in the GCEB, as demanded in section 3.3. All states contributing to S correspond *precisely* to the sector with K units of longitudinal momentum, and not to a range (as was criticized in section 3.4). The light-cone frame is ideal for the use of hamiltonian methods and computation of bound states, and discrete light-cone quantization facilitates this task further [37].

5. Discussion

We have achieved our goal of obtaining a precise formulation of the Bekenstein bound which also satisfies several formal constraints related to its origin from bounds on light-sheets. We were motivated by the expectation that Bekenstein's bound captures constraints that the holographic principle imposes on the physics of flat space — a point of view that will be discussed in detail in a forthcoming publication [9]. In this section, we note that the DLCQ form of Bekenstein's bound is empirically viable. We also point out some implications and puzzles arising from the null compactification.

5.1 Validity

We expect that the Bekenstein bound in DLCQ form, eq. (4.21), is valid for realistic field theories. Many explicit calculations of spectra in DLCQ have been carried out (see ref. [37] for a review), especially in the context of QCD. In a preliminary survey, we have found no results which contradict eq. (4.21). It will be an interesting task to check the bound systematically against existing results and to calculate more spectra for further verification.

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Because of the rapidly increasing complexity of diagonalizing the hamiltonian, results in the literature pertain mostly to small values of K, but this is the most interesting range in any case. When a large number of quanta is present, the bound tends to be easily satisfied [40]. Violations of the bound would require a surprisingly strong growth of the number of bound states with K, at low K.

The species problem, which appeared to be resolved by interactions [26], resurfaces in the DLCQ form. One can write down lagrangians that populate the $K = 1$ sector with an arbitrary number Q of fundamental one-particle states. Unless the theory is confining, the bound will thus be violated if $Q > \exp(2\pi^2) \approx 3 \times 10^8$. We interpret this as a prediction that lagrangians with such a large number of fields are not consistent with quantum gravity. Certainly there are no indications that such lagrangians would be realistic.

Before we took the step of null compactification, the restriction to bound states followed automatically from the requirement of finite spatial width. Now, however, it must be imposed explicitly. Particles can scatter off to infinity in the uncompactified transverse dimensions. Scattering states contribute a continuous part to the spectrum, which must be ignored when calculating the entropy. An interesting question, which we do not investigate here, is whether long-lived resonances can be treated in a controlled way. Even the proton is probably metastable, not to speak of ordinary macroscopic systems, to which the bound ought to apply nevertheless. It may turn out that such states are effectively included because they have stable antecedents at finite K where the resolution does not suffice to describe the decay products.

Our prescription has a further restriction which, one hopes, can be relaxed without sacrificing precision: that the transverse spatial dimensions are noncompact. One would like to consider not only exact Minkowski space (with the required null identification), but also compactifications from higher-dimensional theories. The resulting tower of Kaluza-Klein modes gives an infinite number of species from the lower-dimensional point of view. If we wish to apply the bound to flat space with compact dimensions, it is natural to restrict to the massless sector. In many string compactifications, this sector can still contain a considerable number of species ($Q \sim 10^4$), but we are not aware of examples which exceed the bound. Another acceptable limit may be to consider only states which are so well localized in the compact dimension that the situation is equivalent to higher-dimensional flat space. However, it is difficult to distinguish such bound states from states which would become unstable in the decompactification limit.

5.2 Implications and puzzles

The precision gained by compactifying a null direction comes at a price. The spectrum in the sectors with small K differs from the true spectrum of the theory, which is strictly recovered only in the decompactification limit $K \to \infty$. At finite K, sufficiently complex systems and fine spectral features are not resolved.

However, DLCQ does approximate physical states with $Ma/\hbar \ll K$ very well [36]. Thus, for sufficiently large K, the DLCQ form does connect with more traditional but less precise formulations of Bekenstein's bound, in which a particular matter system with fixed mass and size is given.

What is somewhat mysterious is whether and how our refinement of the entropy definition lifts back to the more general light-sheets allowed by the covariant entropy bound. In the weak gravity limit, the specific light-sheets chosen for the derivation of Bekenstein's bound become a null hyperplane ($x + t = \mathrm{const}$). But generically, the cross-sectional area of light-sheets decreases. Such light-sheets cannot be periodically identified along the null direction. It may be more useful to think of DLCQ as an imposition of periodicity rather than the physical compactification of light-rays.

The appearence of DLCQ when making Bekenstein's bound precise may indicate that this form of quantization plays a preferred role in the emergence of ordinary flat space physics from an underlying quantum gravity theory (just as null hypersurfaces may have a special significance in how general relativity arises). If this is the case, we will eventually discover a physical interpretation of the spectra at finite K.

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